

# Univalent categories and the Rezk completion

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## 3 kinds of sameness for categories

<b>Equality</b>	$C = D$
<b>Isomorphism</b>	$C \cong D$
<b>Equivalence</b>	$C \simeq D$

- most properties of categories invariant under **equivalence**
- we can only substitute **equals for equals**
- in set-theoretic foundations these notions are worlds apart

In this talk:

Define categories in the **Univalent Foundations** for which all three coincide

- 1 Introduction to Univalent Foundations
- 2 Category Theory in Univalent Foundations

# Univalent Foundations

## What are the Univalent Foundations?

- Intensional Martin-Löf Type Theory
- ↪ *Types as Spaces* interpretation, i.e. Homotopy Type Theory
- + **Univalence Axiom**

## Martin-Löf Type Theory and its Homotopy Interpretation

Sigma type      $\sum_{x:A} B(x)$      total space of a fibration

Product type      $\prod_{x:A} B(x)$      space of sections of a fibration

Identity type      $\text{Id}_A(a, b)$      space of **paths**  $p : a \rightsquigarrow b$

# Univalence : equivalent types are equal

## Universes in MLTT

- Types in MLTT are stratified in **Universes**  $\mathcal{U}_n$
- can consider  $\text{Id}_{\mathcal{U}}(A, B)$  (polymorphic in universe level  $n$ )
- **Univalence** allows to construct identities between  $A$  and  $B$

## Univalence

- Define type  $\text{Equiv}(A, B)$  of **Equivalences from  $A$  to  $B$**
- Univalence Axiom identifies  $\text{Equiv}(A, B)$  with  $\text{Id}(A, B)$
- Can construct  $f : \text{Equiv}(A, B)$  for suitable  $A, B$

# Level of a type

## Definition (Propositions & Sets)

A type  $A$  is a **proposition** if any two  $a, b : A$  are equal, that is,

$$\text{isProp}(A) := \prod_{x, y : A} \text{Id}(x, y)$$

A type  $A$  is a **set** if for any  $x, y : A$ , the type  $\text{Id}(x, y)$  is a proposition

$$\text{isSet}(A) := \prod_{x, y : A} \text{isProp}(\text{Id}(x, y))$$

- Propositions are “proof–irrelevant” types.
- Points of a set are equal in a unique way, if they are.

# Equivalence of types

## Definition (Equivalence of types)

A function  $f : A \rightarrow B$  is an **equivalence of types** if there are

- 

$$g : B \rightarrow A$$

- 

$$\eta : \prod_{a:A} \text{Id}(g(f(a)), a) \quad \epsilon : \prod_{b:B} \text{Id}(f(g(b)), b)$$

together with a coherence condition  $\tau : \prod_{x:A} \text{Id}(f(\eta x), \epsilon(fx))$

- “ $f$  is an equivalence” is a proposition, written  $\text{isEquiv}(f)$

- 

$$\text{Equiv}(A, B) := \sum_{f:A \rightarrow B} \text{isEquiv}(f)$$

# The Univalence Axiom

Definition (From paths to equivalences)

$$\begin{aligned} \text{id\_to\_equiv}_{A,B} &: \text{Id}(A, B) \rightarrow \text{Equiv}(A, B) \\ \text{refl}(A) &\mapsto (a \mapsto a) \end{aligned}$$

Univalence Axiom

$$\text{univalence} : \prod_{A, B: \mathcal{U}} \text{isEquiv}(\text{id\_to\_equiv}_{A,B})$$

In particular, Univalence gives a map backwards:

$$\text{equiv\_to\_id}_{A,B} : \text{Equiv}(A, B) \rightarrow \text{Id}(A, B)$$



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## Notation

Write  $p : x \rightsquigarrow y$  for  $p : \text{Id}_A(x, y)$

# Categories in Univalent Foundations — Take I

## A naïve definition of categories

A **category**  $\mathcal{C}$  is given by

- a type  $\mathcal{C}_0$  of **objects**
- for any  $a, b : \mathcal{C}_0$ , a type  $\mathcal{C}(a, b)$  of **morphisms**
- operations: identity & composition

$$\text{id}_a : A(a, a)$$

$$(\circ)_{a,b,c} : A(b, c) \rightarrow A(a, b) \rightarrow A(a, c)$$

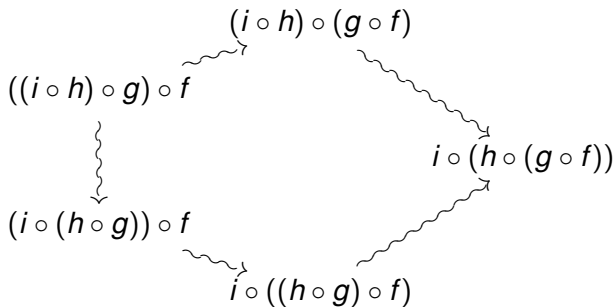
- axioms: unitality & associativity

$$\text{id} \circ f \rightsquigarrow f \quad f \circ \text{id} \rightsquigarrow f \quad (h \circ g) \circ f \rightsquigarrow h \circ (g \circ f)$$

**Problem:** Would require higher coherence data...

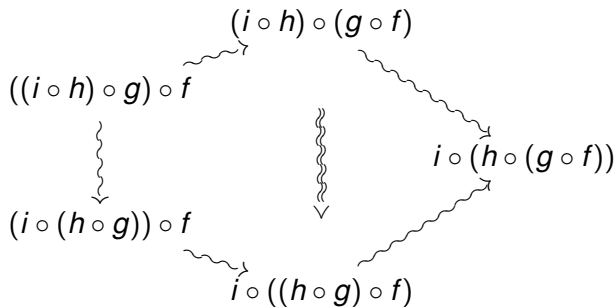
# Coherence for associativity

Two ways to associate a composition of **four** morphisms from left to right:



# Coherence for associativity

Two ways to associate a composition of **four** morphisms from left to right:



Would need to ask for higher coherence  $\rightsquigarrow$  ,  $\rightsquigarrow\rightsquigarrow$  etc

# Categories in Univalent Foundations — Take II

## A less naïve definition of categories

A **category**  $\mathcal{C}$  is given by

- a type  $\mathcal{C}_0$  of objects
- for any  $a, b : \mathcal{C}_0$ , a `set`  $\mathcal{C}(a, b)$  of morphisms
- operations: identity & composition
- axioms: unitality & associativity

For this definition of category, the pentagon is automatically coherent.

# Isomorphism in a category

## Definition (Isomorphism in a category)

A morphism  $f : \mathcal{C}(a, b)$  is an **isomorphism** if there are

- 

$$g : \mathcal{C}(b, a)$$

- 

$$\eta : g \circ f \rightsquigarrow \text{id}_a \quad \epsilon : f \circ g \rightsquigarrow \text{id}_b$$

- “ $f$  is an isomorphism” is a proposition, written  $\text{isIso}(f)$

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$$\text{Iso}(a, b) := \sum_{f:\mathcal{C}(a,b)} \text{isIso}(f)$$

# From paths to isomorphisms

Definition (From paths to isomorphisms, univalent categories)

For objects  $a, b : \mathcal{C}_0$  we define

$$\begin{aligned} \text{id\_to\_iso}_{a,b} : (a \rightsquigarrow b) &\rightarrow \text{Iso}(a, b) \\ \text{refl}(a) &\mapsto \text{id}_a \end{aligned}$$

We call the category  $\mathcal{C}$  **univalent** if, for any objects  $a, b : \mathcal{C}_0$ ,

$$\text{id\_to\_iso}_{a,b} : (a \rightsquigarrow b) \rightarrow \text{Iso}(a, b)$$

is an equivalence of types.

- In a univalent category, isomorphic objects are equal.
- “ $\mathcal{C}$  is univalent” is a proposition, written  $\text{isUniv}(\mathcal{C})$ .



# Examples of univalent categories

- SET (follows from the Univalence Axiom)
- categories of algebraic structures (groups, rings,...)
  - made precise by the **Structure Identity Principle** (Coquand, Aczel)
- full subcategories of univalent categories
- functor category  $\mathcal{D}^{\mathcal{C}}$ , if  $\mathcal{D}$  is univalent (see below)

# What about categories as objects?

## Definition (Functor)

A **functor**  $F$  from  $\mathcal{C}$  to  $\mathcal{D}$  is given by

- a map  $F_0 : \mathcal{C}_0 \rightarrow \mathcal{D}_0$
- for any  $a, a' : \mathcal{C}_0$ , a map  $F_{a,a'} : \mathcal{C}(a, a') \rightarrow \mathcal{D}(Fa, Fa')$
- preserving identity and composition

A functor  $F$  is an **isomorphism of categories** if

- $F_0$  is an equivalence of types
- $F_{a,a'}$  is an equivalence of types (a bijection) for any  $a, a'$

$$\mathcal{C} \cong \mathcal{D} := \sum_{F:\mathcal{C}\rightarrow\mathcal{D}} \text{isIsoOfCats}(F)$$

# Natural transformations

## Definition (Natural transformation)

Let  $F, G : \mathcal{C} \rightarrow \mathcal{D}$  be functors. A **natural transformation**  $\alpha : F \rightarrow G$  is given by

- for any  $a : \mathcal{C}_0$  a morphism  $\alpha_a : \mathcal{D}(Fa, Ga)$  s.t.
- for any  $f : \mathcal{C}(a, b)$ ,  $Gf \circ \alpha_a \rightsquigarrow \alpha_b \circ Ff$

The type of natural transformations  $F \rightarrow G$  is a **set**.

## Definition (Functor category $\mathcal{D}^{\mathcal{C}}$ )

- objects: functors from  $\mathcal{C}$  to  $\mathcal{D}$
- morphisms from  $F$  to  $G$ : natural transformations

A natural transformation  $\alpha$  is an isomorphism iff each  $\alpha_a$  is.

# Equivalence of categories

## Definition (Left Adjoint, Equivalence of Categories)

A functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  is a **left adjoint** if there are

- $G : \mathcal{D} \rightarrow \mathcal{C}$
- $\eta : 1_{\mathcal{C}} \rightarrow GF$
- $\epsilon : FG \rightarrow 1_{\mathcal{D}}$
- + higher coherence data.

A left adjoint  $F$  is an **equivalence of categories** if  $\eta$  and  $\epsilon$  are isomorphisms.

*“ $F$  is an equivalence”* is a proposition.

$$\mathcal{C} \simeq \mathcal{D} := \sum_{F:\mathcal{C}\rightarrow\mathcal{D}} \text{isEquivOfCats}(F)$$

# 1 kind of sameness for univalent categories

<b>Equality</b>	$\mathcal{C} \rightsquigarrow \mathcal{D}$
<b>Isomorphism</b>	$\mathcal{C} \cong \mathcal{D}$
<b>Equivalence</b>	$\mathcal{C} \simeq \mathcal{D}$

## Theorem

For **univalent** categories  $\mathcal{C}$  and  $\mathcal{D}$ , these three are equivalent as types.

In particular, we can substitute a univalent category with an equivalent one.

# Rezk completion

- “Being univalent” is a proposition
- ↪ Inclusion from univalent categories to categories

## Theorem

*The inclusion of univalent categories into categories has a left adjoint (in bicategorical sense),*

$$\mathcal{C} \mapsto \widehat{\mathcal{C}} \quad \textbf{Rezk completion of } \mathcal{C}$$

That is, any functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  with  $\mathcal{D}$  univalent factors uniquely via  $\eta_{\mathcal{C}} : \mathcal{C} \rightarrow \widehat{\mathcal{C}}$ :

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\eta_{\mathcal{C}}} & \widehat{\mathcal{C}} \\ & \searrow \exists & \vdots \exists! \\ & & \mathcal{D} \end{array}$$

$\eta$  is unit of adjunction

# Formalization and reference

## Formalization in Coq

- Rezk completion formalized
- approx. 4000 lines of code
- based on Voevodsky's library "*Foundations*"

↪ [github.com/benediktahrens/rezk\\_completion](https://github.com/benediktahrens/rezk_completion)

## References

- preprint with same title [arxiv.org/abs/1303.0584](https://arxiv.org/abs/1303.0584)
- C. Rezk, *A model for the homotopy theory of homotopy theory*, 2001