Coinitial semantics for redecoration of triangular matrices

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Goal

- Category-theoretic semantics of **heterogeneous** **co**inductive data types in Martin-Löf type theory

$\rightsquigarrow$ Develop a notion of “coalgebra” for the signature of a codata type

- Incorporate canonical cosubstitution

In this talk we consider

the codata type of **infinite triangular matrices**

Outline

1. Syntax: initiality for W-types and heterogeneous inductives
2. Cosyntax: infinite triangular matrices and redecoration
3. Coinitiality for Tri

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Starting point: W-types in Martin-Löf TT

Well-founded trees as initial algebra of polynomial functor

- Trees specified by A and $x : A \vdash B(x)$
- Type of natural numbers $\mathbb{N} := W\ A\ B$ with:

$$A = \{O, S\}, \quad B(O) = \{\}, \quad B(S) = \{*\}$$

- The type $W\ A\ B$ is the (carrier of) the initial algebra of

$$X \mapsto \sum_{a:A} X^{B(a)}$$

- Dybjer '97, Moerdijk and Palmgren '00

Heterogeneous data types

motivation: **binding syntax**

$$\begin{aligned}\mathsf{LC}(X : \mathsf{Type}) \;\; ::= \;\; & \mathsf{Var} : X \to \mathsf{LC}(X) \\ \mid \;\; & \mathsf{App} : \mathsf{LC}(X) \times \mathsf{LC}(X) \to \mathsf{LC}(X) \\ \mid \;\; & \mathsf{Abs} : \mathsf{LC}(X+1) \to \mathsf{LC}(X)\end{aligned}$$

heterogeneity of Abs:

recursive argument with bigger parameter $X + 1$

substitution:

$$\mathsf{subst}_{X,Y} : \big(X \to \mathsf{LC}(Y)\big) \to \mathsf{LC}(X) \to \mathsf{LC}(Y)$$

avoiding capture:

$$\mathsf{shift}_{X,Y} : \big(X \to \mathsf{LC}(Y)\big) \to X + 1 \to \mathsf{LC}(Y + 1)$$

Initial semantics for binding syntax

Initial semantics for lambda calculus: Fiore, Plotkin & Turi '99

- characterizes not only data type but also **substitution**
- reformulated using **monads** by Hirschowitz & Maggesi '07

Basis for this reformulation:

Lemma (Substitution is monadic: Altenkirch & Reus '99)

$(\mathsf{LC}, \mathsf{Var}, \mathsf{subst})$ *forms a monad (in Kleisli form)*

Initial semantics for λ-calculus using monads

Definition (Algebra for signature of LC, H & M '07)

- a monad $(T, \mathsf{unit}, \gg=)$ on Type
- two **morphisms of modules over** T,

$$App : T \times T \to T$$
$$Abs : T(\_ + 1) \to T$$

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"Module morphism" expresses **distributivity of substitution**:

$$App\ (s, t) \gg\!= f \quad = \quad App\ (s \gg\!= f, t \gg\!= f)$$
$$Abs\ t \gg\!= f \quad = \quad Abs\ (t \gg\!= \mathsf{shift}^T\ f)$$

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Lemma (Initial semantics for LC, H & M '07)

$(\mathsf{LC}, \mathsf{App}, \mathsf{Abs})$ *is the initial algebra, where* $\mathsf{LC} = (\mathsf{LC}, \mathsf{Var}, \mathsf{subst})$

Goal: characterize **co**data types with **co**substitution

Goal

dualize techniques of H & M to characterize

- **co**data types with
- **co**substitution

as **terminal** object

In this talk we consider

- the codata type of infinite triangular matrices
- with **redecoration** operation

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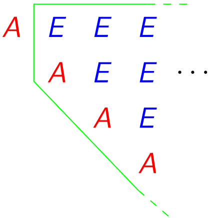
An example of cosyntax: infinite triangular matrices

Tri: the codata type of infinite triangular matrices

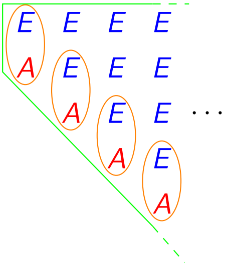
- omit redundant information below the diagonal
- have a **variable** type A of diagonal elements
  - e.g. invertible elements
- a fixed type E of elements for rest of matrix
- usage: Pascal matrices (binomial coefficients), mathematical physics (infinite-dim. problems)

$$\begin{matrix} A & E & E & E & \\ & A & E & E & \cdots \\ & & A & E & \\ & & & A & \end{matrix}$$

$$\frac{t : \mathsf{Tri}(A)}{\mathsf{top}_A(t) : A}$$



$$\frac{t : \mathsf{Tri}(A)}{\mathsf{rest}_A(t) : \mathsf{Tri}(E \times A)}$$



Matrices through trapezia: the destructors of Tri

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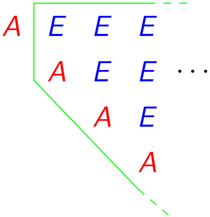

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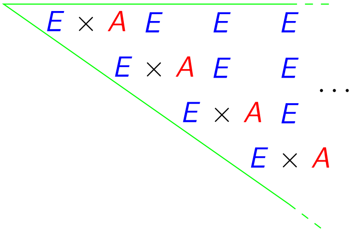
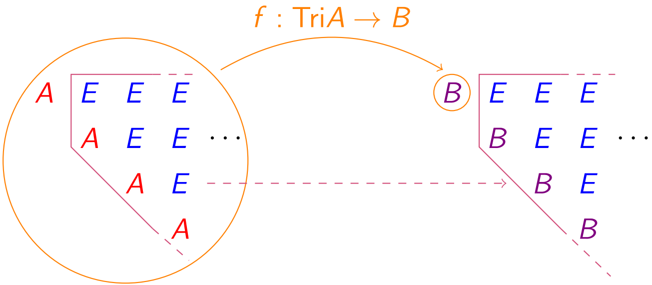


Redecoration

$$\mathsf{redec}_{A,B} : (\mathsf{Tri}A \to B) \to (\mathsf{Tri}A \to \mathsf{Tri}B)$$



$$\mathsf{top}\big(\mathsf{redec}\ f\ \ t\big) := f\ \ t \quad \text{and}$$
$$\mathsf{rest}\big(\mathsf{redec}\ f\ \ t\big) := \mathsf{redec}\big(\mathsf{lift}\ f\big)(\mathsf{rest}\ t)\ \ .$$

with $\mathsf{lift}\ f : \mathsf{Tri}(E \times A) \to E \times B$

Tri is a weak constructive comonad

Sameness = bisimilarity

Bisimilarity $\sim$ coinductively defined via destructors

$$\frac{t \sim t'}{\mathsf{top}(t) = \mathsf{top}(t')} \qquad \frac{t \sim t'}{\mathsf{rest}(t) \sim \mathsf{rest}(t')}$$

Lemma (Matthes and Picard '11)

$(\mathsf{Tri} : \mathsf{Type} \to \mathsf{Type}, \mathsf{top}, \mathsf{redec})$ *forms a "weak constructive comonad".*

- $\rightsquigarrow$ "weak constructive" refers to compatibility conditions with bisimilarity

Tri is a relative comonad

Alternatively, $\mathsf{Tri}A$ is a **setoid** rather than a (plain) type

$$\begin{aligned} \mathsf{top}_A &: \mathsf{Setoid}(\mathsf{Tri}A, \mathsf{eq}A) \\ \mathsf{redec}_{A,B} &: \mathsf{Setoid}(\mathsf{Tri}A, \mathsf{eq}B) \to \mathsf{Setoid}(\mathsf{Tri}A, \mathsf{Tri}B) \end{aligned}$$

with $\mathsf{eq} : \mathsf{Type} \to \mathsf{Setoid}$

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Lemma (Reformulation of Matthes and Picard '11)

$(\mathsf{Tri} : \mathsf{Type} \to \mathsf{Setoid}, \mathsf{top}, \mathsf{redec})$ *forms a* ***comonad relative to*** $\mathsf{eq} : \mathsf{Type} \to \mathsf{Setoid}$.

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Definition (Relative (co)monad, Alten., Chapm. & Uust. '10)

- underlying functor is **not** necessarily **endo**
- needs "mediating" functor (above: eq)

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Towards "coalgebras" for the signature of Tri

Goal: define "coalgebra" s.t. Tri is (the terminal) one

More specifically:

- define a notion of "morphism" for destructor rest
- requirement: want to capture interplay of rest and redec:

$$\mathsf{rest}\big(\mathsf{redec}\ f\ \ t\big) = \mathsf{redec}\big(\mathsf{lift}\ f\big)(\mathsf{rest}\ \ t)$$

Analogous situation for syntax:

For the lambda calculus with monadic substitution:

$$\mathsf{subst}\ f\ \ (\mathsf{Abs}\ \ t) = \mathsf{Abs}\ (\mathsf{subst}\ (\mathsf{shift}\ \ f)\ \ t)$$

(Co)modules over (relative) (co)monads

Morphisms of modules over monads

characterize distributivity of substitution over constructors

$$\mathsf{Abs} : \mathsf{LC}(\_ + 1) \to \mathsf{LC}$$

$$\mathsf{subst}\ f\ (\mathsf{Abs}\ t) = \mathsf{Abs}\ (\mathsf{subst}\ (\mathsf{shift}\ f)\ t)$$

Morphisms of **co**modules over **relative co**monads

characterize distributivity of **co**substitution over destructors

$$\mathsf{rest} : \mathsf{Tri} \to \mathsf{Tri}(E \times \_)$$

$$\mathsf{rest}\ (\mathsf{redec}\ f\ t) = \mathsf{redec}\ (\mathsf{lift}\ f)\ (\mathsf{rest}\ t)$$

Coalgebras for the signature of Tri

Definition (Category of coalgebras)

A coalgebra for the signature of Tri is given by a pair (T, r):

- a comonad T relative to $\mathsf{eq} : \mathsf{Type} \to \mathsf{Setoid}$
- a morphism of comodules over T

$$r : T \to T(E \times \_)$$

Morphisms: …

Lemma

$(\mathsf{Tri}, \mathsf{rest})$ *is the terminal object in the above category.*

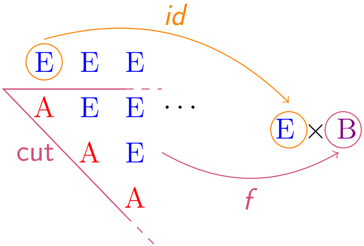
That's **almost** how it works …

Technical difficulty: definition of lift

Definition of

$$\mathsf{lift}_A : (\mathsf{Tri} A \to B) \to \mathsf{Tri}(E \times A) \to E \times B$$

requires auxiliary function $\mathsf{cut}_A : \mathsf{Tri}(E \times A) \to \mathsf{Tri} A$



A specified cut for any coalgebra

- we were not able to define cut categorically
- fix: every coalgebra (T, r) comes with a **specified**

$$c_A : T(E \times A) \to TA$$

  and equations characterizing c
- $c^{\mathsf{Tri}} :=$ cut for Tri uniquely determined by these equations

Lemma (for real this time)

$(\mathsf{Tri}, \mathsf{cut}, \mathsf{rest})$ *is terminal in the category of coalgebras* (T, c, r).

Formalization

Mechanization in the proof assistant `Coq`

- helped to find a mistake we made (already made by Abel, Matthes & Uustalu '05)
- reuses code by Matthes & Picard '11
- 3000 lines of code, among which 1500 by MP '11
- fully constructive and axiom-free
- tedious: coinduction in `Coq` cumbersome

Summary

- Ad hoc notion of weak constructive comonad is instance of relative comonad
- Develop comodules over relative comonads
- Terminal algebra semantics for codata family `Tri`
- Bisimilarity and redecoration are part of universal object
- Not as straightforward as the λ-calculus because of `cut`
- Mechanization in `Coq`

Summary

- Ad hoc notion of weak constructive comonad is instance of relative comonad
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Thanks for your attention

Some references

Some references

- Altenkirch, Chapman & Uustalu: *Monads need not be endofunctors*
- Hirschowitz & Maggesi: *Modules over Monads and Linearity*
- Matthes & Picard: *Verification of Redecoration for Infinite Triangular Matrices using Coinduction*
- preprint about this work on the arXiv

TikZ pictures used with permission from Matthes and Picard