A modular formalization of bicategories in type theory

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Outline

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1. Why bicategories?
2. What is a bicategory?
3. Formalization of bicategories in type theory

Motivation: signatures for type theories

Our overall goal

To give a general notion of signature for type theories, by mutual induction:

- Model of empty signature
- Arity α over signature Σ given by some structure on the models of Σ
- Model of signature (Σ, α) is a pair of a model of Σ together with an instance of the structure specified by α

Models of the empty signature

Many options for what models of empty signature should be:

Form a 1-category:

- C-systems a.k.a. contextual categories

Form a bicategory:

- Categories with Families (CwFs)
- Categories with Attributes
- . . .

- In this project, we opt for CwFs, for their explicitness.
- Want to formalize signatures, since they are complicated objects

⇝ Got sidetracked into this formalization project

Outline

Categories

A category $\mathscr{C}$ is

- a type $\mathscr{C}_0$ of *objects*
- for any two objects $a, b : \mathscr{C}_0$, a **set** $\mathscr{C}(a,b)$ of *morphisms*
- composition function: $(\circ) : \mathscr{C}(b,c) \times \mathscr{C}(a,b) \longrightarrow \mathscr{C}(a,c)$
- for any $a : \mathscr{C}_0$, an identity $1_a : \mathscr{C}(a,a)$
- associativity
- left and right unitality, e.g.,

$$1_b \circ f = f$$

Examples

- sets and functions
- groups and group homomorphisms
- vector spaces and linear maps
- C-systems and their maps

What about the category of categories?

A **functor** F **from** $\mathscr{C}$ **to** $\mathscr{D}$ is given by

- a map $F_o : \mathscr{C}_o \to \mathscr{D}_o$ between the objects
- a family of maps $F_{a,b} : \mathscr{C}(a,b) \to \mathscr{D}(F_o a, F_o b)$

preserving compositions and identities.

- Cats and functors between them not naturally a cat
- Morphisms of functors: natural transformations
- Functors between two fixed cats, and natural transformations between them, form a cat

Examples of bicategories

- Categories, functors, and natural transformations
- Categories with families, structure-preserving functors, good natural transformations

Categories

A category is

- a type $\mathscr{C}_o$ of objects
- for any two objects $a, b : \mathscr{C}_o$, a set $\mathscr{C}(a, b)$
- composition function

$$(\circ) : \mathscr{C}(b, c) \times \mathscr{C}(a, b) \to \mathscr{C}(a, c)$$

- for any $a : \mathscr{C}_o$, an identity $1_a : 1 \to \mathscr{C}(a, a)$
- associativity
- left and right unitality, e.g.,

$$1_b \circ f = f$$

Bicategories: approach I

A bicategory is

- a type $\mathscr{C}_0$ of objects
- for any two objects $a, b : \mathscr{C}_0$, a category $\mathscr{C}(a,b)$
- composition functor

$$(\circ) : \mathscr{C}(b,c) \times \mathscr{C}(a,b) \to \mathscr{C}(a,c)$$

- for any $a : \mathscr{C}_0$, an identity functor $1_a : 1 \to \mathscr{C}(a,a)$
- associator
- left and right unitor, e.g.,

$$1_b \circ f \quad \cong_2 \quad f$$

- triangle and pentagon laws

Bicategories: approach II

A bunch of data, starting with 0-cells, 1-cells, and 2-cells:

- a type $\mathscr{C}_0$ of 0-cells
- for any two 0-cells $a, b : \mathscr{C}_0$, a type $\mathscr{C}_1(a, b)$ of 1-cells
- for any two (parallel) 1-cells $f, g : \mathscr{C}_1(a, b)$, a set $\mathscr{C}_2(f, g)$ of 2-cells
- 9 operations
  - composition and identity for 1-cells
  - unitor and associator 2-cells
- 16 laws

Definition is . . .

- more verbose, and difficult to check for a human
- easier to use in a proof assistant, and suitable for building bicategories in layers

Examples of bicategories

- Bicategory of categories
- Bicategory of “categories with structure” for models of type theory:
  - categories with families
  - categories with attributes
  - categories with display maps
- A monoidal category can be seen as a bicategory with one object
- Rewrite systems give rise to a bicategory:
  - 0-cells: types $A, B, \ldots$
  - 1-cells $f : A_1 \times \ldots \times A_n \to B$: terms of type B in context $\vec{A}$.
  - 2-cells: rewriting rules between (parallel) terms
- Bicategory of topological spaces, continuous maps, and homotopies between them. (This is actually a higher category truncated to a bicategory.)

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Displayed categories

Back to (1-)categories...

- Algebraic structures are built by layering data and properties. E.g., a group is a monoid with inverses.
- Extends to morphisms between structures, e.g., a group homomorphism is a monoid homomorphism preserving inverses (automatic in this particular case).

To summarize: categories of algebraic structures are built by layering. The layers are **displayed categories**.

Building categories in layers

Types	Categories
Base type A	Base category $\mathscr{C}$
Dependent type $B : A \to \mathsf{type}$	Displayed category $\mathscr{D}$ over $\mathscr{C}$
Sigma type $\sum_{x:A} B(x)$	Total category $\int_{\mathscr{C}} \mathscr{D}$
Projection function $\sum_{x:A} B(x) \to A$	Projection functor $\int_{\mathscr{C}} \mathscr{D} \to \mathscr{C}$

An important difference

'Transport' for displayed categories not necessarily available: notion of (bi)fibration

Displayed 1-categories

Given a category $\mathscr{C}$, a **displayed category** $\mathscr{D}$ **over** $\mathscr{C}$ consists of

- for each $c : \mathscr{C}$, a type $\mathscr{D}_c$
- for each $f : \mathscr{C}(a, b)$ of $\mathscr{C}$ and $x : \mathscr{D}_a$ and $y : \mathscr{D}_b$, a **set** $\mathscr{D}_f(x, y)$
- for all $f : a \to b$ and $g : b \to c$ in $\mathscr{C}$ and $x : \mathscr{D}_a$ and $y : \mathscr{D}_b$ and $z : \mathscr{D}_c$, a function

$$(\circ) : \mathscr{D}_g(y, z) \times \mathscr{D}_f(x, y) \to \mathscr{D}_{g \circ f}(x, z),$$

- for each $c : \mathscr{C}$ and $x : \mathscr{D}_c$, a morphism $1_x : \mathscr{D}_{1_c}(x, x)$
- **laws—well-typed modulo axioms of** $\mathscr{C}$

Displayed bicategories

- Displayed bicategory is analogous to displayed category
- Expressed naturally in approach II, not so naturally in I

Examples of displayed bicategories

- Presheaves (over bicat of categories):
  - Objects over a category $\mathscr{C}$: presheaves on $\mathscr{C}$
  - Morphisms over a functor $F : \mathscr{C} \to \mathscr{D}$ from P to P': natural transformations $P \Longrightarrow P'F^{op}$
  - 2-cells over natural transformation $\alpha : F \Longrightarrow F'$: equality of natural transformations
- Displayed 1-categories (over bicat of categories):
  - Objects over a category $\mathscr{C}$: displayed 1-categories over $\mathscr{C}$
  - Morphisms over a functor $F : \mathscr{C} \to \mathscr{D}$ from $\mathscr{D}$ to $\mathscr{D}'$: displayed 1-functors
  - 2-cells over natural transformation $\alpha : F \Longrightarrow F'$: ...

Categories with families

A **category with families** consists of:

1. a category $\mathscr{C}$, together with
2. presheaves $\mathsf{Ty}, \mathsf{Tm} : \mathscr{C}^{op} \to \mathsf{Set}$;
3. a natural transformation $\mathsf{p} : \mathsf{Tm} \to \mathsf{Ty}$; and
4. for each object $\Gamma : \mathscr{C}$ and $A : \mathsf{Ty}(\Gamma)$, a representation of the fiber of p over A, i.e.

   4.1 ...
   4.2 ...
   4.3 ...

We have built the bicategory of CwFs by stacking displayed bicategories.

Conclusions

Work done so far

- Maps back and forth to bicats (I) (by Lumsdaine and Riley)
- Displayed bicategory of displayed 1-categories
- Various displayed bicategories over bicategory of categories
- Bicategory of CwFs as total bicat of iterated displayed category
- Pseudofunctors: homomorphisms of bicategories
- Proof: left and right unitor coincide on identity

Constructions in progress

- Univalence for bicategories; example: the univalent bicategory of univalent categories
- (Strict) Fibre bicategory of a displayed bicategory (seems only possible under some assumptions)

Conclusions II

Code

- About 5000 loc so far
- Available on `https://github.com/UniMath/UniMath`, in subdir `UniMath/CategoryTheory/Bicategories`

Thanks a lot!