Univalent Foundations and the equivalence principle

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Indiscernability of identicals

Identical objects satisfy the same properties

$$x = y \;\; \rightarrow \;\; \forall P \, (P(x) \leftrightarrow P(y))$$

- Reasoning **in logic** is invariant under equality
- **In mathematics**, reasoning should be invariant under weaker notion of sameness!

The equivalence principle

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Equivalence principle

Reasoning in mathematics should be **invariant under** the appropriate notion of **sameness**.

- **equal** numbers, functions,...
- **isomorphic** sets, groups, rings,...
- **equivalent** categories
- **biequivalent** bicategories
- ...

Violating the equivalence principle

Can **violate** this principle:

Exercise

Find a statement about categories that is not invariant under the equivalence of categories

$$\bullet \rightleftarrows \bullet \quad \simeq \quad \bullet$$

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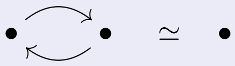

A solution

“The category $\mathcal{C}$ has exactly one object.”

A language for invariant properties

M. Makkai, *Towards a Categorical Foundation of Mathematics*:

> *The basic character of the Principle of Isomorphism is that of a* ***constraint on the language*** *of Abstract Mathematics; a welcome one, since it provides for the separation of sense from nonsense.*

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Goal

to have a **syntactic criterion** for properties and constructions that are invariant under equivalence

How to break the invariance principle for categories…

- Recall: the statement

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  is not invariant under equivalence of categories.
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  The category $\mathcal{C}$ has exactly one object.

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- Referring to **equality of objects** breaks invariance, but…
- even the **definition** of category refers to equality of objects

...and how to fix it.

Solution

Use a logic/language of **dependent types**, in which $s(g) = t(f)$ is encoded by what type of thing f and g are.

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A category consists of

- a collection O of objects
- for each $x, y \in O$, a collection $A(x, y)$ of arrows
- for each $x, y, z \in O$ and each $f \in A(x, y)$ and $g \in A(y, z)$, a composite $g \circ f \in A(x, z)$
- for each $x \in O$, an identity $\mathsf{id}_x \in A(x, x)$
- ...

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Gives rise to **dependently typed language** by adding logical connectors.

Invariance for properties

Theorem (Freyd '76, Blanc '78)

A property of categories (expressed in 2-sorted first order logic) is invariant under equivalence iff it can be expressed in this dependently typed language, using equality for arrows but not for objects.

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- What about **constructions** on categories?

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- What about **constructions** on categories?
- What about other mathematical structures?

Equivalence principle in the univalent foundations

In the univalent foundations

- an equivalence principle can be proved for a variety of structures
  - sets
  - groups, rings, . . .
  - categories
- EP applies not only to properties, but also to constructions.

Transport

For a given dependent type

$$x : A \ \vdash \ B(x)$$

and $a, b : A$, the rules of the identity type allow to construct a term

$$\mathsf{transport}^B : B(a) \times (a = b) \to B(b)$$

Contractible types, propositions and sets

- A is **contractible** if we can construct a term of type

$$\mathsf{isContr}(A) \stackrel{\text{def}}{=} \sum_{(x:A)} \prod_{(y:A)} y = x$$

- A is a **proposition** if

$$\mathsf{isProp}(A) \stackrel{\text{def}}{=} \prod_{x,y:A} \mathsf{isContr}(x = y)$$

- A is a **set** if

$$\mathsf{isSet}(A) \stackrel{\text{def}}{=} \prod_{x,y:A} \mathsf{isProp}(x = y)$$

$$\mathsf{Prop} \stackrel{\text{def}}{=} \sum_{x:\mathsf{U}} \mathsf{isProp}(X) \qquad \mathsf{Set} \stackrel{\text{def}}{=} \sum_{X:\mathsf{U}} \mathsf{isSet}(X)$$

Equivalences

Definition

A map $f : A \to B$ is an **equivalence** if it has contractible fibers, i.e.,

$$\mathsf{isequiv}(f) \stackrel{\text{def}}{=} \prod_{b:B} \mathsf{isContr}\left(\sum_{a:A} f(a) = b\right)$$

The type of equivalences:

$$A \simeq B \stackrel{\text{def}}{=} \sum_{f:A \to B} \mathsf{isequiv}(f)$$

Characterizing some identity types

Can construct equivalences

- for $f, g : A \to B$

$$\big(f = g\big) \simeq \Big(\prod_{a:A} f(a) = g(a)\Big)$$

- for $s, t : A \times B$

$$\big(s = t\big) \simeq \Big(\big(\mathsf{pr}_1(s) = \mathsf{pr}_1(t)\big) \times \big(\mathsf{pr}_2(s) = \mathsf{pr}_2(t)\big)\Big)$$

- for $s, t : \sum_{(x:A)} B(x)$

$$\big(s = t\big) \simeq \left(\sum_{e:\mathsf{pr}_1(s)=\mathsf{pr}_1(t)} \mathsf{transport}^B\big(e,\, \mathsf{pr}_2(s)\big) = \mathsf{pr}_2(t)\right)$$

Voevodsky's Univalence Axiom

Naïve answer

$$\mathsf{univalence} : (A =_{\mathsf{U}} B) \simeq (A \simeq B)$$

More controlled:

Answer

Define

$$\mathsf{idtoeqv} : \prod_{A,B:\mathsf{U}} (A = B) \to (A \simeq B)$$

$$\mathsf{refl}_A \mapsto \mathsf{id}$$

$$\text{Axiom } \mathsf{univalence} : \prod_{A,B:\mathsf{U}} \mathsf{isequiv}(\mathsf{idtoeqv}_{A,B})$$

Invariance under equivalence

For a given predicate $P : \mathsf{U} \to \mathsf{U}$ and $A, B : \mathsf{U}$, from

$$\mathsf{transport}^P : P(A) \times (A = B) \to P(B)$$

and

$$(A =_{\mathsf{U}} B) \simeq (A \simeq B)$$

obtain

$$P(A) \times (A \simeq B) \to P(B)$$

Groups in Univalent Foundations

A **group** $G = (X, m, i, e)$ in Univalent Foundations is

- a set X
- operations

$$\begin{array}{ccccc} & & X \times X & & \\ & & \downarrow{\scriptstyle m} & & \\ X & \xrightarrow{i} & X & \xleftarrow{e} & 1 \end{array}$$

- such that group axioms are satisfied

The type of groups is

$$\mathsf{Grp} := \sum_{X:\mathsf{Set}} \mathsf{GrpStructure}(X)$$

Lifting univalence from types to groups

A **group isomorphism** $G \to G'$ is

- a bijective function on the underlying types $X \to X'$
- compatible with the group structures (m, i, e) and (m', i', e') on X and X'.

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Theorem (EP on types lifts to EP on groups)

An isomorphism of groups lifts to an equivalence of all constructions on groups (in UF):

$$\prod_{(P:\mathsf{Grp}\to\mathsf{U})} \prod_{(G,G':\mathsf{Grp})} (G \cong G') \times P(G) \to P(G')$$

Lifting univalence from types to groups

The proof of this statement uses 2 ingredients:

1. $(G = G') \;\simeq\; (G \cong G')$
2. Transport along identities

$(G = G') \;\simeq\; (G \cong G')$ is given by the canonical map

$$\mathsf{refl}_G \mapsto \mathsf{id}_G$$

Identity is isomorphism for groups

$$
\begin{aligned}
G = G' &\simeq (X,S) = (X',S') \\
&\simeq \sum_{p:X=X'} \mathsf{transport}^{\mathsf{GrpStructure}}(p,S) = S' \\
&\simeq \sum_{p:X=X'} (\mathsf{transport}^{Y\mapsto(Y\times Y\to Y)}(p,m) = m') \\
&\qquad \times (\mathsf{transport}^{Y\mapsto(Y\to Y)}(p,i) = i') \\
&\qquad \times (\mathsf{transport}^{Y\mapsto(1\to Y)}(p,e) = e') \\
&\simeq \sum_{f:X\simeq X'} \left(f\circ m\circ(f^{-1}\times f^{-1}) = m'\right) \\
&\qquad \times \left(f\circ i\circ f^{-1} = i'\right) \\
&\qquad \times (f\circ e = e') \\
&\simeq (G \cong G')
\end{aligned}
$$

Lifting univalence to algebraic structures

Lifting univalence to algebraic structures (Aczel, Coquand, Danielsson)

For many algebraic structures in univalent foundations, univalence lifts.

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Examples:

- rings
- posets
- discrete fields
- sets with fixpoint operator

This general result is best explained in terms of categories.

Categories in univalent type theory

A category is

- a type $O : \mathsf{U}$ of objects
- a dependent type $A : O \times O \to \mathcal{S}et$ of arrows
- $\mathsf{id} : \prod_{(a:O)} A(a,a)$
- $(\circ) : \prod_{(a,b,c:O)} A(a,b) \times A(b,c) \to A(a,c)$
- axioms postulating identities of arrows

Categories in univalent type theory

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- $(\circ) : \prod_{(a,b,c:O)} A(a,b) \times A(b,c) \to A(a,c)$
- axioms postulating identities of arrows
- such that the natural map

$$\mathsf{idtoiso} : \prod_{a,b:O} (a = b) \to \mathsf{iso}(a,b)$$

is an equivalence for any a, b

Examples of univalent categories

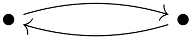
- $\mathcal{S}et$ (discrete types)
- Groups, rings, posets,… (Structure Identity Principle)
- Functor category $[\mathcal{C}, \mathcal{D}]$, if $\mathcal{D}$ is univalent
- Full subcategories of univalent categories

More examples of univalent categories

- A preorder is univalent iff it is antisymmetric
- If X is of h-level 3, i.e., 1-truncated, then there is a univalent category with X as objects and $\hom(x, y) := (x = y)$
- If $\mathcal{C}$ is univalent, then the category of cones of shape $F : \mathcal{J} \to \mathcal{C}$ is

Non-univalent categories

- Any "chaotic" category $\mathcal{C}$ with $\mathcal{C}(x, y) := 1$, for $\mathcal{C}_0$ not a prop



- Any chaotic category $\mathcal{C}$ with an object $c : \mathcal{C}_0$ is **equivalent** to the terminal category $\mathbf{1}$ (which is univalent)

Univalence for categories

Univalence for algebraic structures

For groups, rings, etc., univalence lifts.

Univalence for categories

Univalence for algebraic structures

For groups, rings, etc., univalence lifts.

Going to equivalence of categories:

Univalence for categories

For **univalent** categories, equivalence is identity via canonical map

$$(\mathcal{C} = \mathcal{D}) \;\simeq\; \mathsf{Equiv}(\mathcal{C}, \mathcal{D}) \;.$$

Summary

- univalence axiom asserts equivalence principle (EP) for types
- EP for types lifts to EP for groups etc.
- EP for higher-categorical structures requires an additional restriction (e.g., EP for **univalent** categories)

Some references

- Freyd, *Properties invariant within equivalence types of categories*
- Blanc, *Equivalence naturelle et formules logiques en théorie des catégories*
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