A Higher Structure Identity Principle

Benedikt Ahrens

j.w.w. Paige R North, Michael Shulman, Dimitris Tsementzis

Motivation

Equivalence principle

Two equivalent structures must share the same structural properties.

Our goal

To define, in univalent foundations, a large class of *structures* and a notion of *equivalence* between them validating the equivalence principle.

- Inspired by *First Order Logic with Dependent Sorts*, Makkai, 1995.
- Generalizing *Univalent categories and the Rezk completion*, Ahrens, Kapulkin, Shulman, 2015.

Overview

1. Equivalence principle for categories
2. FOLDS signature and structures for categories
3. Digression: equality and theories
4. Isomorphisms and univalence
5. Equivalence principle for FOLDS structures

Outline

1. Equivalence principle for categories
2. FOLDS signature and structures for categories
3. Digression: equality and theories
4. Isomorphisms and univalence
5. Equivalence principle for FOLDS structures

Categories in type theory

A **category** $\mathscr{C}$ is given by

- a type $\mathscr{C}_0 : \mathscr{U}$ of **objects**
- for any $a, b : \mathscr{C}_0$, a set $\mathscr{C}(a, b) : \mathscr{U}$ of **morphisms**
- operations: identity & composition

$$1_a : \mathscr{C}(a, a)$$
$$(\circ)_{a,b,c} : \mathscr{C}(b, c) \to \mathscr{C}(a, b) \to \mathscr{C}(a, c)$$

- axioms: unitality & associativity

$$1 \circ f = f \qquad f \circ 1 = f \qquad (h \circ g) \circ f = h \circ (g \circ f)$$

Categories in type theory

A **category** $\mathscr{C}$ is given by

- a type $\mathscr{C}_0 : \mathscr{U}$ of **objects**
- for any $a, b : \mathscr{C}_0$, a set $\mathscr{C}(a, b) : \mathscr{U}$ of **morphisms**
- operations: identity & composition

$$1_a : \mathscr{C}(a, a)$$
$$(\circ)_{a,b,c} : \mathscr{C}(b, c) \to \mathscr{C}(a, b) \to \mathscr{C}(a, c)$$

- axioms: unitality & associativity

$$1 \circ f = f \qquad f \circ 1 = f \qquad (h \circ g) \circ f = h \circ (g \circ f)$$

A **univalent category** is a category $\mathscr{C}$ such that

$$(a = b) \to (a \cong b)$$

is an equivalence for all $a, b : \mathscr{C}_0$.

Local univalence implies global univalence

Theorem

For categories A and B, let $A \simeq B$ denote the type of equivalences from A to B. If A and B are univalent, we have

$$(A =_{\mathsf{UCat}} B) = (A \simeq B).$$

Corollary

If A and B are equivalent univalent categories, then they share the same properties.
For any $X : \mathsf{UCat} \vdash P(X) : \mathscr{U}$,

$$(A \simeq B) \to (P(A) = P(B)).$$

Goal

Prove a similar theorem for other categorical structures.

Envisioned result

Given a signature $\mathscr{L}$, and two $\mathscr{L}$-univalent $\mathscr{L}$-structures M and N, then

$$(M = N) = (M \simeq_{\mathscr{L}} N)$$

Need notions of

- signatures $\mathscr{L}$
- $\mathscr{L}$-structures
- $\mathscr{L}$-equivalence of $\mathscr{L}$-structures
- $\mathscr{L}$-univalence of $\mathscr{L}$-structures

Outline

1. Equivalence principle for categories
2. FOLDS signature and structures for categories
3. Digression: equality and theories
4. Isomorphisms and univalence
5. Equivalence principle for FOLDS structures

Two-level type theory

Working in the two-level type theory of Annenkov-Capriotti-Kraus.

- Universes $\mathscr{U} \hookrightarrow \mathscr{U}^s$
- $\mathscr{U}$ implements univalent type theory.
- Every type $T : \mathscr{U}^s$ is equipped with a strict equality type $a \equiv_T b$ with the usual rules for the identity type, but which also satisfies UIP.

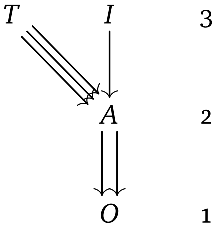
FOLDS-signatures

Definition

A signature is a graded one-way semicategory of finite height:

- all arrows go “downwards”
- the height is the minimum natural number with non-empty type of objects

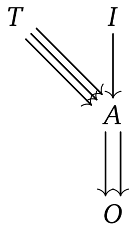
$\mathscr{L}_{\text{cat}} :=$



with some strict equalities between morphisms

Overview: $\mathscr{L}_{\text{cat}}$-structures

We can define the data of a category $\mathscr{C}$ to be



- A type $\mathscr{C}O : \mathscr{U}$
- A family $\mathscr{C}A : \mathscr{C}O \times \mathscr{C}O \to \mathscr{U}$
- A family $\mathscr{C}I : \prod_{(x:\mathscr{C}O)} \mathscr{C}A(x,x) \to \mathscr{U}$
- A family $\mathscr{C}T : \prod_{(x,y,z:\mathscr{C}O)} \mathscr{C}A(x,y) \to \mathscr{C}A(y,z) \to \mathscr{C}A(x,z) \to \mathscr{U}$

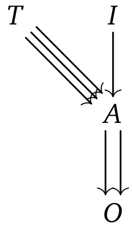
Here:

- Think of $\mathscr{C}I$, $\mathscr{C}T$ as the *predicates* 'is an identity', 'is a composite'.
- $\mathscr{L}_{\text{cat}}$-*univalence* will imply that $\mathscr{C}I$, $\mathscr{C}T$ are pointwise propositions.
- $\mathscr{L}_{\text{cat}}$-*univalence* will imply that $\mathscr{C}A$ is pointwise a set.
- $\mathscr{L}_{\text{cat}}$-*univalence* will imply that $\mathscr{C}O$ is a 1-type.

Overview: $\mathscr{L}_{\text{cat}}$-structures

We can define the data of a category $\mathscr{C}$ to be

- A type $\mathscr{C}O : \mathscr{U}$
- A family $\mathscr{C}A : \mathscr{C}O \times \mathscr{C}O \to \mathscr{U}$
- A family $\mathscr{C}I : \prod_{(x:\mathscr{C}O)} \mathscr{C}A(x,x) \to \mathscr{U}$
- A family $\mathscr{C}T : \prod_{(x,y,z:\mathscr{C}O)} \mathscr{C}A(x,y) \to \mathscr{C}A(y,z) \to \mathscr{C}A(x,z) \to \mathscr{U}$



Remark: these $\mathscr{L}_{\text{cat}}$-structures do not satisfy any axioms, e.g.,

- composition is a functional relation
- left and right unitality for composition

Such axioms will be discussed now.

Outline

1 Equivalence principle for categories

2 FOLDS signature and structures for categories

3 Digression: equality and theories

4 Isomorphisms and univalence

5 Equivalence principle for FOLDS structures

Equality

To the data, we add axioms such as

- "There is a composite of every composable pair of arrows."

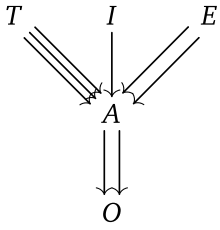
$$\forall(x,y,z:O).\forall(f:A(x,y)).\forall(g:A(y,z)).\exists(h:A(x,z)).T_{x,y,z}(f,g,h)$$

- "Composites are unique."

$$\forall(x,y,z:O).\forall(f:A(x,y)).\forall(g:A(y,z)).\forall(h,h':A(x,z)).\\ T_{x,y,z}(f,g,h)\to T_{x,y,z}(f,g,h')\to(h=h')$$

So we need to add an equality 'predicate':

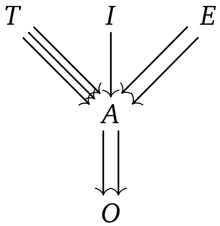
$\mathscr{L}_{\mathrm{cat+E}} :=$



$\mathscr{L}_{\text{cat+E}}$-structures

We can define the data of a category $\mathscr{C}$ to be

- A type $\mathscr{C}O : \mathscr{U}$
- A family $\mathscr{C}A : \mathscr{C}O \times \mathscr{C}O \to \mathscr{U}$
- A family $\mathscr{C}I : \prod_{(x:\mathscr{C}O)} \mathscr{C}A(x,x) \to \mathscr{U}$
- A family $\mathscr{C}T : \prod_{(x,y,z:\mathscr{C}O)} \mathscr{C}A(x,y) \to \mathscr{C}A(y,z) \to \mathscr{C}A(x,z) \to \mathscr{U}$
- A family $\mathscr{C}E : \prod_{(x,y:\mathscr{C}O)} \mathscr{C}A(x,y) \to \mathscr{C}A(x,y) \to \mathscr{U}$



Here:

- $\mathscr{L}_{\text{cat+E}}$-*univalence* will imply that $\mathscr{C}E$ is a proposition.
- $\mathscr{L}_{\text{cat+E}}$-*univalence* + axioms making E into an equivalence relation and congruence will imply that $(f = g) = \mathscr{C}E(f,g)$.

1-univalent FOLDS-categories

A 1-univalent FOLDS-category consists of an $\mathscr{L}_{\mathrm{cat+E}}$-structure

- $\mathscr{C}O : \mathscr{U}$
- $\mathscr{C}A : \mathscr{C}O \times \mathscr{C}O \to \mathscr{U}$
- $\mathscr{C}I : \prod_{(x:\mathscr{C}O)} \mathscr{C}A(x,x) \to \mathscr{U}$
- $\mathscr{C}T : \prod_{(x,y,z:\mathscr{C}O)} \mathscr{C}A(x,y) \to \mathscr{C}A(y,z) \to \mathscr{C}A(x,z) \to \mathscr{U}$
- $\mathscr{C}E : \prod_{(x,y:\mathscr{C}O)} \mathscr{C}A(x,y) \to \mathscr{C}A(x,y) \to \mathscr{U}$

such that

- $\mathscr{C}I_x(f)$, $\mathscr{C}T_{x,y,z}(f,g,h)$, and $\mathscr{C}E_{x,y}(f,g)$ are propositions
- $\mathscr{C}A(x,y)$ is a set,
- $\mathscr{C}E_{x,y}(f,g) = (f = g)$,

and the axioms of a category are satisfied.

Lemma

The type of 1-univalent FOLDS-cats is equivalent to the type of (pre)categories.

Summary: equality and axioms

Summary on equality

- Can add equality predicate
- Typically add equality on top-level (e.g., for A, but not for O)
- Imposing suitable axioms on equality predicate ensures it is equivalent to actual equality

Remark

The notion of isomorphism and univalence we give does not depend on axioms for structures, but only on the shape of structures, i.e., on the signature

Outline

1. Equivalence principle for categories
2. FOLDS signature and structures for categories
3. Digression: equality and theories
4. Isomorphisms and univalence
5. Equivalence principle for FOLDS structures

Univalent FOLDS-categories

Goal

To state the univalence condition

$$(a = b) = (a \cong b)$$

for categories in terms of the the FOLDS structure.

Given $a, b : \mathscr{C}O$, we can define an isomorphism $a \cong b$ using the Yoneda Lemma:

- For each $x : \mathscr{C}O$, an equality $\phi_{x\bullet} : \mathscr{C}A(x,a) = \mathscr{C}A(x,b)$.
- For each $x,y : \mathscr{C}O$, $f : \mathscr{C}A(x,y)$, $g : \mathscr{C}A(y,a)$, and $h : \mathscr{C}A(x,a)$, we have

$$\mathscr{C}T_{x,y,a}(f,g,h) = \mathscr{C}T_{x,y,b}(f, \phi_{y\bullet}(g), \phi_{x\bullet}(h))$$

with $\phi_{y\bullet}(g) \circ f = \phi_{x\bullet}(g \circ f)$

This is a bit ad hoc and not symmetric.

FOLDS isomorphism for categories

Instead, can define $a \cong b$ to consist of the following equalities between all the types of our signature with a and b substituted in *all* possible ways:

- For each $x : \mathscr{C}O$, an equality $\phi_{x\bullet} : \mathscr{C}A(x,a) = \mathscr{C}A(x,b)$.
- For each $z : \mathscr{C}O$, an equality $\phi_{\bullet z} : \mathscr{C}A(a,z) = \mathscr{C}A(b,z)$.
- An equality $\phi_{\bullet\bullet} : \mathscr{C}A(a,a) = \mathscr{C}A(b,b)$.
- The following equalities for all appropriate w, x, y, z, f, g, h:

$$T_{x,y,a}(f,g,h) = T_{x,y,b}(f, \phi_{y\bullet}(g), \phi_{x\bullet}(h))$$
$$T_{x,a,z}(f,g,h) = T_{x,b,z}(\phi_{x\bullet}(f), \phi_{\bullet z}(g), h)$$
$$T_{a,z,w}(f,g,h) = T_{b,z,w}(\phi_{\bullet z}(f), g, \phi_{\bullet w}(h))$$
$$T_{x,a,a}(f,g,h) = T_{x,b,b}(\phi_{x\bullet}(f), \phi_{\bullet\bullet}(g), \phi_{x\bullet}(h))$$
$$T_{a,x,a}(f,g,h) = T_{b,x,b}(\phi_{\bullet x}(f), \phi_{x\bullet}(g), \phi_{\bullet\bullet}(h))$$
$$T_{a,a,x}(f,g,h) = T_{b,b,x}(\phi_{\bullet\bullet}(f), \phi_{\bullet x}(g), \phi_{\bullet x}(h))$$
$$T_{a,a,a}(f,g,h) = T_{b,b,b}(\phi_{\bullet\bullet}(f), \phi_{\bullet\bullet}(g), \phi_{\bullet\bullet}(h))$$

$$I_{a,a}(f) = I_{b,b}(\phi_{\bullet\bullet}(f))$$
$$E_{x,a}(f,g) = E_{x,b}(\phi_{x\bullet}(f), \phi_{x\bullet}(g))$$
$$E_{a,x}(f,g) = E_{b,x}(\phi_{\bullet x}(f), \phi_{\bullet x}(g))$$
$$E_{a,a}(f,g) = E_{b,b}(\phi_{\bullet\bullet}(f), \phi_{\bullet\bullet}(g))$$

"Everything above a, b thinks that a and b are the same."

Univalent FOLDS categories

Theorem

In any 1-univalent FOLDS category, the type of isomorphisms $a \cong b$ just defined is equivalent to the type of ordinary isomorphisms $a \cong b$.

Definition

A univalent FOLDS category is a 1-univalent FOLDS category such that for all $a, b : \mathscr{C}O$, the canonical map

$$(a = b) \to (a \cong b)$$

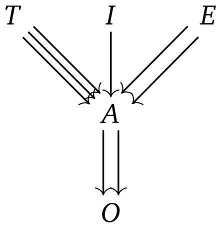
is an equivalence.

Theorem

A 1-univalent FOLDS category is univalent if and only if its corresponding precategory is a univalent category.

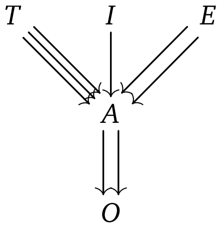
Univalence at O, at A, at T, I, and E

- For an $\mathcal{L}_{\text{cat+E}}$-structure $\mathscr{C}$, we have given a definition of isomorphism for two objects $a, b : \mathscr{C}\,O$.
- This definition does not depend on the axioms of a category (e.g., composition is a function), but only on the signature.



Univalence at O, at A, at T, I, and E

- For an $\mathscr{L}_{\text{cat+E}}$-structure $\mathscr{C}$, we have given a definition of isomorphism for two objects $a, b : \mathscr{C}O$.
- This definition does not depend on the axioms of a category (e.g., composition is a function), but only on the signature.

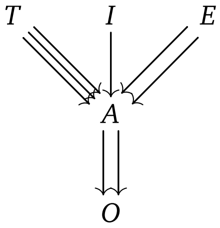


The same principle gives a definition of isomorphism between

- two morphisms $f, g : \mathscr{C}A$ between same objects
- two triangles $c, d : \mathscr{C}T$ in the same fiber, etc.

Univalence at T

- For any $c, d : \mathscr{C}T_{x,y,a}(f, g, h)$, everything above c, d "thinks" c and d are the same, trivially.
- So $(c \cong d) = 1$, and $\mathscr{C}T$ being univalent means that $(c = d) = (c \cong d)$.
- Thus, $\mathscr{C}$ being univalent at T means that each $\mathscr{C}T_{x,y,a}(f, g, h)$ is a proposition.
- Similar for I and E



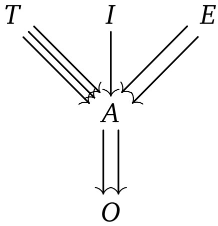
Univalence at A

Let $\mathscr{C}$ be a $\mathscr{L}_{\text{cat}}$-structure that is univalent at T, I, and E.

- For any $f, g : \mathscr{C}A(a,b)$, the type $f \cong g$ is a product of equivalences

$$(f \cong g) = \big(T(f,k,h) = T(g,k,h)\big) \times \ldots$$

- These types of equivalences are propositions.
- Thus, $\mathscr{C}$ being univalent at A means that each $\mathscr{C}A(a,b)$ is a set.



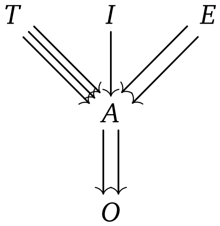
Univalence at A

Let $\mathscr{C}$ be a $\mathscr{L}_{\text{cat}}$-structure that is univalent at T, I, and E.

- For any $f,g : \mathscr{C}A(a,b)$, the type $f \cong g$ is a product of equivalences

$$(f \cong g) = \Big(T(f,k,h) = T(g,k,h)\Big) \times \dots$$

- These types of equivalences are propositions.
- Thus, $\mathscr{C}$ being univalent at A means that each $\mathscr{C}A(a,b)$ is a set.
  - Also, $f \cong g$ implies $E_{a,b}(f,g)$ and conversely, hence univalence at A means $(f = g) = E_{a,b}(f,g)$.



Categorical equivalences

For univalent FOLDS categories $\mathscr{C}, \mathscr{D}$, we had an equivalence.

$$(\mathscr{C} = \mathscr{D}) \simeq (\mathscr{C} \simeq \mathscr{D})$$

We can also FOLDS-ify categorical equivalences:

- A *very split surjective morphism* $F : \mathscr{C} \twoheadrightarrow \mathscr{D}$ of $\mathscr{L}_{\mathrm{cat+E}}$-structures consists of split surjections
  - $F_O : \mathscr{C}O \twoheadrightarrow \mathscr{D}O$
  - $F_A : \prod_{x,y:\mathscr{C}O} \mathscr{C}A(x,y) \twoheadrightarrow \mathscr{D}A(F_O x, F_O y)$
  - $F_T : \prod_{x,y,z:\mathscr{C}O, f:\mathscr{C}A(x,y), g:\mathscr{C}A(y,z), h:\mathscr{C}A(x,z)} \mathscr{C}T(f,g,h) \twoheadrightarrow \mathscr{D}T(F_A f, F_A g, F_A h)$
  - ...

Theorem

For univalent FOLDS categories $\mathscr{C}$ and $\mathscr{D}$ we have

$$(\mathscr{C} \twoheadrightarrow \mathscr{D}) \simeq (\mathscr{C} \simeq \mathscr{D})$$

Outline

1. Equivalence principle for categories
2. FOLDS signature and structures for categories
3. Digression: equality and theories
4. Isomorphisms and univalence
5. Equivalence principle for FOLDS structures

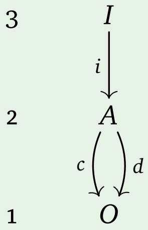
Reminder: signatures

Definition

A signature is a graded one-way semicategory of finite height:

- all arrows go “downwards”
- the height is the minimum natural number with non-empty type of objects

Example (Signature for reflexive graphs)



Derivation by example

$$\begin{array}{cc} 3 & I \\ & i\big\downarrow \\ 2 & A \\ & c\big\downarrow\;\big\downarrow d \\ 1 & O \end{array}$$

Example

In $\mathscr{L}_{\mathrm{rg}}$ we have $(\mathscr{L}_{\mathrm{rg}})_0 \equiv \{O\}$. Let M_0 be a (a function picking out) the two-element set $\{a, b\}$. Then $(\mathscr{L}_{\mathrm{rg}})'_{M_0}$ is the following signature, with four sorts of rank 0 and two sorts of rank 1:

$$\begin{array}{cccc} I(a,a) & & & I(b,b) \\ \big\downarrow i & & & \big\downarrow i \\ A(a,a) & A(a,b) & A(b,a) & A(b,b) \end{array}$$

Structures of a signature

Example

An $\mathscr{L}_{\mathbf{rg}}$**-structure** is given by

- a type M_{o}
- recursively, a structure for the signature $(\mathscr{L}_{\mathrm{rg}})'_{M_{\mathrm{o}}}$

Definition

An $\mathscr{L}$**-structure** M is given by

- a function $M_{\mathrm{o}} : \mathscr{L}_{\mathrm{o}} \to \mathscr{U}$
- recursively, a structure M' for the signature $(\mathscr{L})'_M$

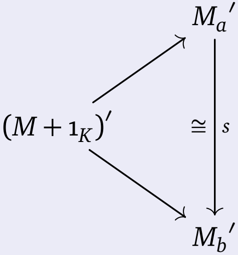
There are also suitable notions of

- morphism of structures
- isomorphism of structures

$\mathscr{L}$-isomorphism

Definition

Let M be an $\mathscr{L}$-structure, $K : \mathscr{L}_0$, and $a, b : MK$. The type $a \cong_M b$ of $\mathscr{L}$-isomorphisms from a to b is defined to be the type of levelwise equivalences $s : M_a \cong {M_b}'$ in $\mathbf{Struc}_{\mathscr{L}'_{M_0+1_K}}$ under $(M + 1_K)'$, i.e. those s such that the following triangle commutes:



$\mathscr{L}$-univalence

Definition ($\mathscr{L}$-univalence)

Let K be an object of $\mathscr{L}_o$. We say that M is **univalent at** K if

$$\prod_{x,y\,:\,MK} \mathsf{isequiv}(\mathsf{idtoiso}^{MK}_{x,y})$$

We say that M is **$\mathscr{L}$-univalent** if

- M is K-univalent for every $K : \mathscr{L}_o$ and
- M' is univalent.

Results

Theorem

Let $\mathscr{L} : \mathsf{Sig}(n+1)$*, M a univalent $\mathscr{L}$-structure, $K : \mathscr{L}_0$. Then $M_0(K)$ is of h-level $n+1$.*

Theorem

For a signature $L : \mathsf{Sig}(n)$*, the type of univalent L-structures is of h-level $n+1$.*

Theorem (Higher Structure Identity Principle)

Consider $\mathscr{L}$-structures M, N for some signature $\mathscr{L}$ such that M is univalent. Then

$$(M = N) = (M \twoheadrightarrow N)$$

Examples

- First-order logic (with equality)
- Categories
- Dagger categories
- (Ana)functors
- Profunctors
- Displayed categories / Fibrations

Remarks

Preprint to be on the arXiv soon

- Two notions of signature: FOLDS-signatures and axiomatic signatures
- Translation from FOLDS- to axiomatic signatures
- Examples in terms of FOLDS-signatures
- Abstract reasoning about axiomatic signatures

Remarks

Preprint to be on the arXiv soon

- Two notions of signature: FOLDS-signatures and axiomatic signatures
- Translation from FOLDS- to axiomatic signatures
- Examples in terms of FOLDS-signatures
- Abstract reasoning about axiomatic signatures

Thanks for your attention!